Exam
SOLID MECHANICS (NASM)
January 19, 2015 (09–12:00)

Question 1 For each of the following statements point out if it is correct or not, *and* why:

a. *Every* continuum solution for stress ($\boldsymbol{\sigma}$), strain ($\boldsymbol{\varepsilon}$) and displacement ($\boldsymbol{u}$) has to satisfy the equations:

$$\operatorname{div} \boldsymbol{\sigma} = \mathbf{0}\,, \tag{1a}$$

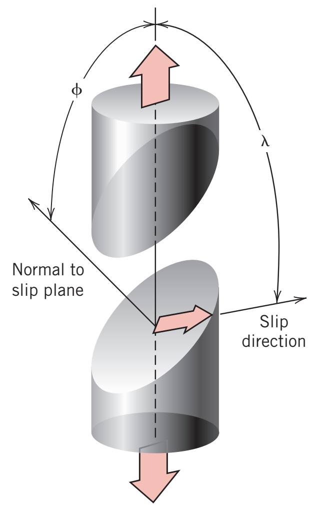
$$\boldsymbol{\varepsilon} = \frac{1}{2}[\operatorname{grad} \boldsymbol{u} + (\operatorname{grad} \boldsymbol{u})^T]\,, \tag{1b}$$

$$\boldsymbol{\sigma} = \mathscr{L}\boldsymbol{\varepsilon}\,. \tag{1c}$$

b. The hydrostatic pressure in an isotropic linear elastic material with a Poisson ratio of $\nu = 1/2$ is always zero.

c. Slip (or glide) in cubic crystals occurs preferentially on planes at 45° from the principal stress direction.

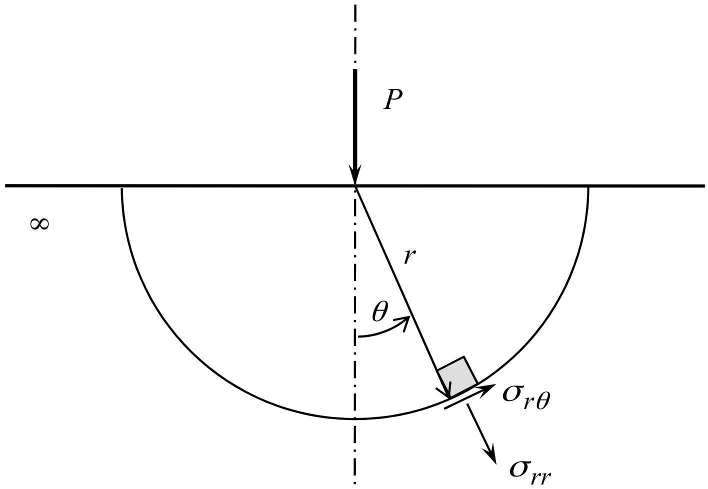
Question 2 In a crystal subjected to uniaxial tension, as indicated in the figure by the vertical arrows, a slip system is identified by the angle ϕ between slip plane normal and tensile direction, and by the angle λ between slip direction and tensile axis.



a. Derive the relation between the applied stress σ and the shear stress resolved on this slip system, τ_r.

b. Derive the analogous relationship when the crystal is subjected to a shear stress τ.

Question 3 One of the classical solutions for tribology is that of Flamant's problem: a half-infinite body in plane strain (normal to the plane of the sketch below) loaded by a point force normal to the free surface. The central question is: what is the stress field? We will consider this in terms of a polar coordinate system (r, θ).



a. Write down all traction boundary conditions along the free surface.

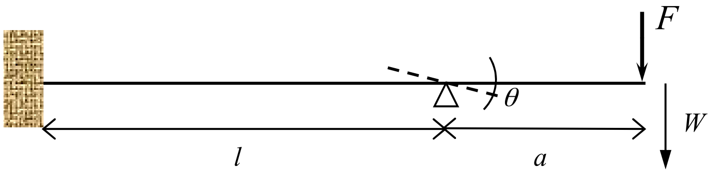
b. Use symmetry to demonstrate that $\sigma_{r\theta}(r, \theta) = 0$.

c. The radial stress field (σ_{rr}) must be such that it is singular at the point force ($r = 0$) and vanishes at $r \to \infty$. A dependence of the type $\sigma_{rr} \propto 1/r$, independent of θ[1], satisfies these boundary conditions. Assuming this, express the radial stress field in terms of the point force P (per unit thickness normal to the plane of consideration). (NB: This can be done solely by considering equilibrium).

d. Determine $\sigma_{\theta\theta}(r, \theta)$ through consideration of the Airy stress function.

[1] This is incorrect!

Question 4 A cantilever of length $l + a$ and bending stiffness EI is simply supported at a distance l and subjected to a load F at the free end.



a. Consider the free-body diagram of the beam only; that is, completely isolated from the environment. Can all the reaction forces and moments from the wall and the support be determined solely by equilibrium? Explain.

b. Specify the boundary conditions.

c. Use the forget-me-nots to determine the tip displacement W and the rotation θ at the support.

✍ ——— ✍

Question	# points
1	1+1+1=3
2	2+1=3
3	1+1+2=4
4	1+1+3=5

Exam grade $= (\#\ \text{points} + 2)/1.7$

Question 1

a. **In**correct: (1c)

b. **In**correct

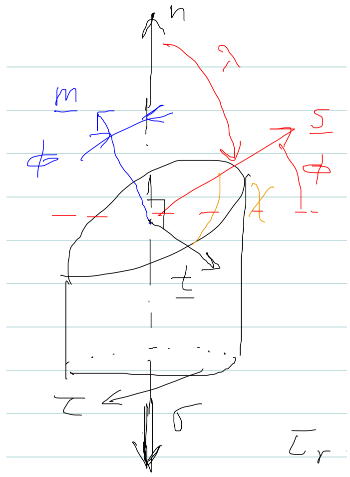
c. **In**correct

Question 2

a. Introduce unit vector in (vertical) tensile direction, $\boldsymbol{n}$.
Then stress tensor is $\boldsymbol{\sigma} = \sigma \boldsymbol{n} \otimes \boldsymbol{n}$ and

$$\begin{aligned}\tau_R &= \boldsymbol{m} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{s} \\ &= \sigma(\boldsymbol{m} \cdot \boldsymbol{n})(\boldsymbol{s} \cdot \boldsymbol{n}) \\ &= \sigma \cos\phi \cos\lambda\end{aligned}$$

b. Shear stress is rather poorly defined as the direction of shear, $\boldsymbol{t}$, is lacking. Let us assume that it is in the plane span by $\boldsymbol{n}$ and $\boldsymbol{s}$ and orthogonal to $\boldsymbol{n}$. Then $\boldsymbol{\sigma} = \tau(\boldsymbol{n} \otimes \boldsymbol{t} + \boldsymbol{t} \otimes \boldsymbol{n})$ and



$$\begin{aligned}\tau_R &= \boldsymbol{m} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{s} \\ &= \tau[(\boldsymbol{m} \cdot \boldsymbol{n})(\boldsymbol{t} \cdot \boldsymbol{s}) + (\boldsymbol{m} \cdot \boldsymbol{t})(\boldsymbol{n} \cdot \boldsymbol{s})] \\ &= \tau[\cos\phi \cos\phi - \sin\phi \cos\lambda]\,.\end{aligned}$$

If, alternatively, the shear stress is applied in the plane span by $\boldsymbol{m}$ and $\boldsymbol{s}$, yet still orthogonal to $\boldsymbol{n}$, then

$$\begin{aligned}\tau_R &= \boldsymbol{m} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{s} \\ &= \tau[(\boldsymbol{m} \cdot \boldsymbol{n})(\boldsymbol{t} \cdot \boldsymbol{s}) + (\boldsymbol{m} \cdot \boldsymbol{t})(\boldsymbol{n} \cdot \boldsymbol{s})] \\ &= \tau[\cos\phi \sin\lambda - \sin\phi \cos\lambda]\,.\end{aligned}$$

Students may have a different sign of τ because they have assumed a 90° degree rotation of the applied shear stress.

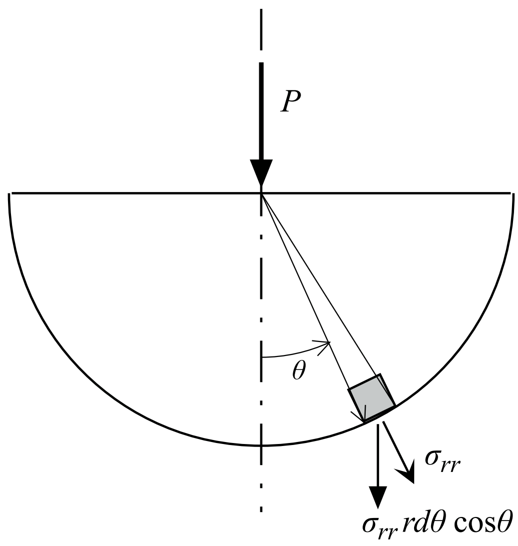
Answers to (a) and (b) can be simplified by noting that since $\phi + \lambda = \pi/2$, $\cos\lambda = \sin\phi$ and $\sin\lambda = \cos\phi$. 2+1

Question 3

a. $\boldsymbol{t} = \boldsymbol{0}$, or $\sigma_{r\theta} = \sigma_{\theta\theta} = 0 \forall r$ except at $r = 0$ 1

b. The problem is symmetric about the vertical z–axis. 1

c. Consider the free-body diagram of a semi-circular region of radius r:



Vertical force equilibrium:

$$P + \int_{-\pi/2}^{\pi/2} \sigma_{rr} \cos\theta r d\theta = 0$$

Substituting $\sigma_{rr} = C/r$, [2] we find $C = -P/2$, hence

$$\sigma_{rr} = -\frac{1}{2}\frac{P}{r}$$

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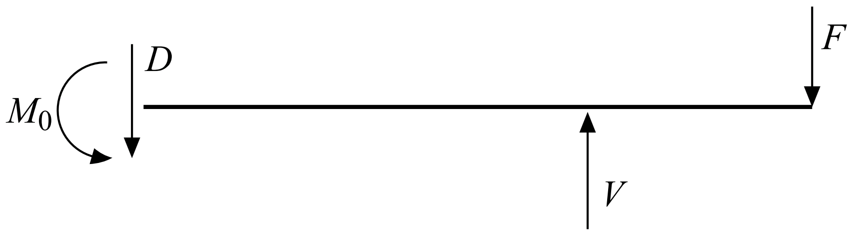
Question 4

a. There are 3 (hidden) unknowns: the reaction forces from the support and the left-hand wall, V and D, plus the bending moment exerted by the wall, M_0. However, there are only two relevant equilibrium conditions: equilibrium in the vertical direction and equilibrium of moments, hence not all unknowns can be determined solely from equilibrium.

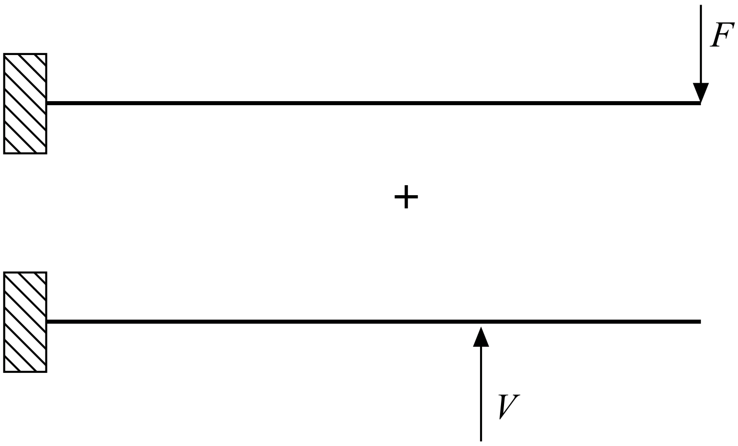


1

b. $w(0) = w(l) = 0$, $w'(0) = 0$. 1

c. Use superposition of the solution for a cantilever:

[2] which is incorrect and should read $\sigma_{rr} = C\cos\theta/r$



By clamping the beam, two of the three BCs of (b) are met, and V is the only unknown force. It is to be determined from the remaining BC that $w(l) = 0$:

$$w(l) = \frac{F}{EI}\left(\frac{1}{2}Ll^2 - \frac{1}{6}l^3\right) - \frac{Vl^3}{3EI},$$

with $L = l + a$. The condition that $w(l) = 0$ gives

$$V = F\left(1 + \frac{3}{2}\frac{a}{l}\right).$$

The tip deflection (downwards) due to V is given by

$$-\frac{Vl^3}{3EI} + \theta_V(l)a$$

with $\theta_V(l) = w_V'(l) = -(Vl^2)/(2EI)$. Hence the total tip deflection becomes

$$\begin{aligned} w(L) &= \frac{FL^3}{3EI} - \frac{Vl^3}{3EI} - \frac{Vl^2a}{2EI} \\ &= \frac{Fa^3}{3EI}\left\{1 + \frac{3}{4}\frac{l}{a}\right\} \end{aligned}$$

The rotation at l due to F is to be found from $w'(x)$ for an end-loaded cantilever:

$$w_F'(x) = \frac{F}{EI}\left(Lx - \frac{1}{2}x^2\right)$$

so in total

$$\begin{aligned} w'(l) &= \frac{F}{EI}\left(Ll - \frac{1}{2}l^2\right) - \frac{Vl^2}{2EI} \\ &= \frac{5Fal}{4EI} \end{aligned}$$

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